A fine scale analysis of spatially adapted total variation regularisation

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Introduction

Motivation: In the celebrated total variation (TV) model for image denoising

$$\min_{u\in \mathrm{BV}(\Omega)} \frac{1}{2}\int_\Omega (f-u)^2dx + \alpha |Du|(\Omega),$$

the regularisation strength is uniform over the image domain Ω, due to the parameter α being a scalar quantity only.

➡ Disadvantageous when both **fine scale details** and **homogeneous areas** are present in the data f. As many contributions in the literature have shown, this problem can be remedied via spatially dependent regularisation, by introducing weights either in the fidelity or in the total variation term:

Weighted TV model

$$\min_{u\in \mathrm{BV}(\Omega)} \frac{1}{2}\int_\Omega (f-u)^2dx + \int_\Omega \boldsymbol{\alpha(x)} d|Du|$$

$$\alpha\in C(\overline{\Omega}),\quad \alpha>0$$

Weighted fidelity model

$$\min_{u\in \mathrm{BV}(\Omega)} \frac{1}{2}\int_\Omega \boldsymbol{w}(f-u)^2dx + |Du|(\Omega)$$

$$w\in L^\infty(\Omega),\quad w\geq 0$$

While the analysis of the regularisation properties of the standard scalar total variation has received a considerable amount of attention, this is not the case for the two models above.

- **This study fills in that gap in the literature through an extended fine scale analysis of the one dimensional versions of the weighted TV and weighted fidelity models.**

Weighted TV - Creation of Jumps

Optimality conditions for the 1D weighted TV problem: Let $\Omega=(a,b)\subseteq\mathbb{R}$, $f\in \mathrm{BV}(\Omega)$, $\alpha\in C(\overline{\Omega})$ with $\alpha>0$. Then $u\in \mathrm{BV}(\Omega)$ solves the weighted TV problem if there exists a function $v\in H_0^1(\Omega)$ such that

$$v'=f-u,$$
$$-v\in \alpha \mathrm{Sgn}(Du),$$

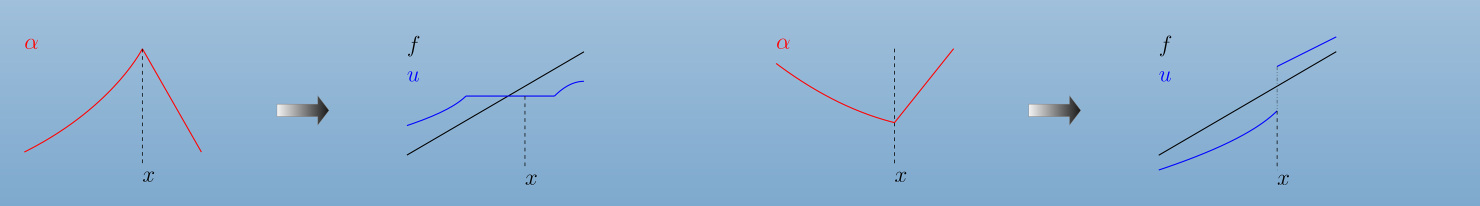
where $\mathrm{Sgn}(Du)=\left\{v\in L^\infty(\Omega)\cap L^\infty(\Omega,|Du|):\ \|v\|_\infty\leq 1,\ v=\frac{Du}{|Du|},\ |Du|\text{-a.e.}\right\}$.

In [4], the author showed that if $\alpha\in W^{1,\infty}(\Omega)$ with $\nabla\alpha\in \mathrm{BV}(\Omega)$, then the solution u of the weighted TV problem, satisfies $J_u\subseteq J_f\cup J_{\nabla\alpha}$, where J denotes the jump set of a BV function. This means that **new jump discontinuities** can potentially be created in places where the gradient of the weight function α has jumps. Using the above optimality conditions, we refine this result in 1D as follows:

Proposition: Let $\Omega=(a,b)\subseteq\mathbb{R}$, $f\in \mathrm{BV}(\Omega)$, $\alpha\in C(\overline{\Omega})$ with $\alpha>0$, $\alpha'\in \mathrm{BV}(\Omega)$, and u be the solution to the weighted TV problem with data f and weight α. Then the following hold:

(i) If $D\alpha'(\{x\})<0$ then u is constant in a neighbourhood of x.
(ii) The estimate $|Du|(\{x\})\leq|Df|(\{x\})+|D\alpha'|(\{x\})$ holds at every $x\in\Omega$.
(iii) If $|Df|(\{x\})=0$, $D\alpha'(\{x\})>0$ and $(x-\epsilon,x+\epsilon)\subseteq \mathrm{supp}(|Du|)$, then $|Du|(\{x\})=D\alpha'(x)$.
(iv) If f and u jump in different directions then $|Du|(\{x\})\leq||Df|(\{x\})-D\alpha'(\{x\})|$.
(v) If $|D\alpha'|(\{x\})=0$ and both u, f jump at x, then their jumps have the same direction.

- Moreover if $\alpha'>2\|f\|_\infty$ in an interval I, then u is constant in I, i.e., also steep weight functions create flat zones in the solution u.

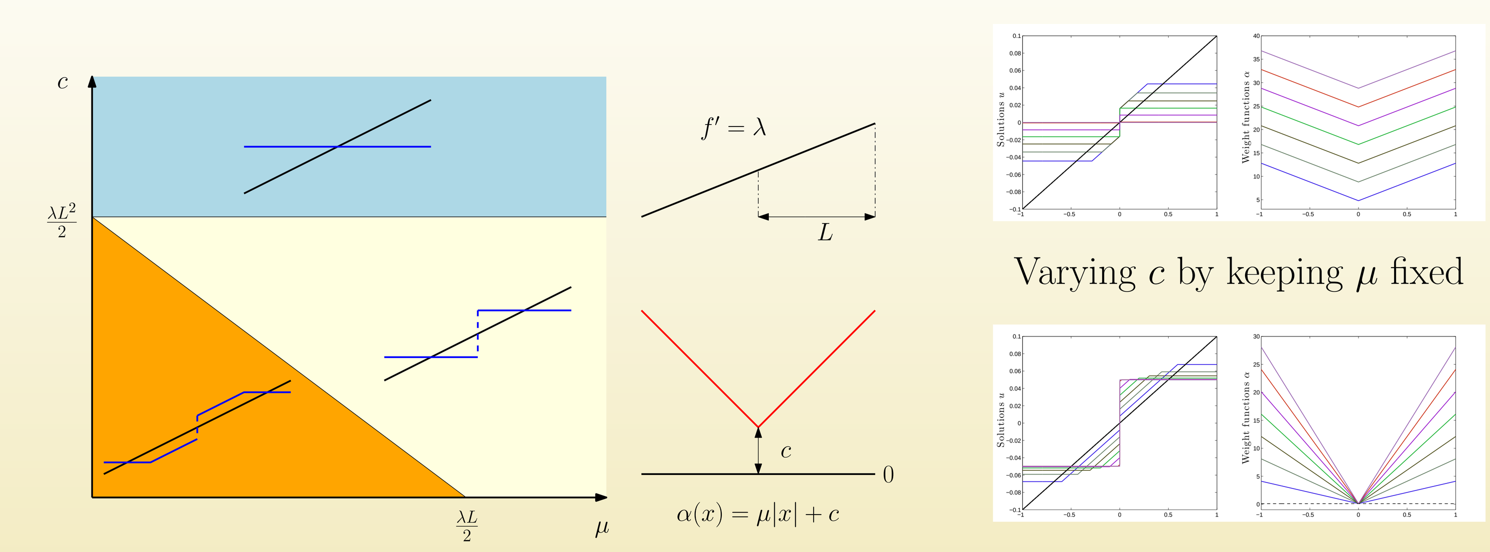


Weight functions α with upward "kinks" ($D\alpha'(\{x\})<0$) always create flat zones in the solution u

Weight functions α with downward "kinks" ($D\alpha'(\{x\})>0$) may create jumps in the solution u

Creation of jumps - Analytically computed example

Setting $\Omega=(-L,L)$, $f(x)=\lambda x$, $\lambda>0$, we have computed analytically all the solutions of the weighted TV problem with data f and weight functions $\alpha(x)=\mu|x|+c$, with $\mu,c>0$:



Varying c by keeping μ fixed

Varying μ by keeping c fixed

Weighted TV and Semigroup Property

Set $S_\alpha(f):=\underset{u\in \mathrm{BV}(\Omega)}{\mathrm{argmin}}\ \frac{1}{2}\int_\Omega(f-u)^2dx+\int_\Omega \alpha(x)d|Du|$. It is a well-known fact that the following *semigroup property* holds for the one dimensional scalar total variation problem

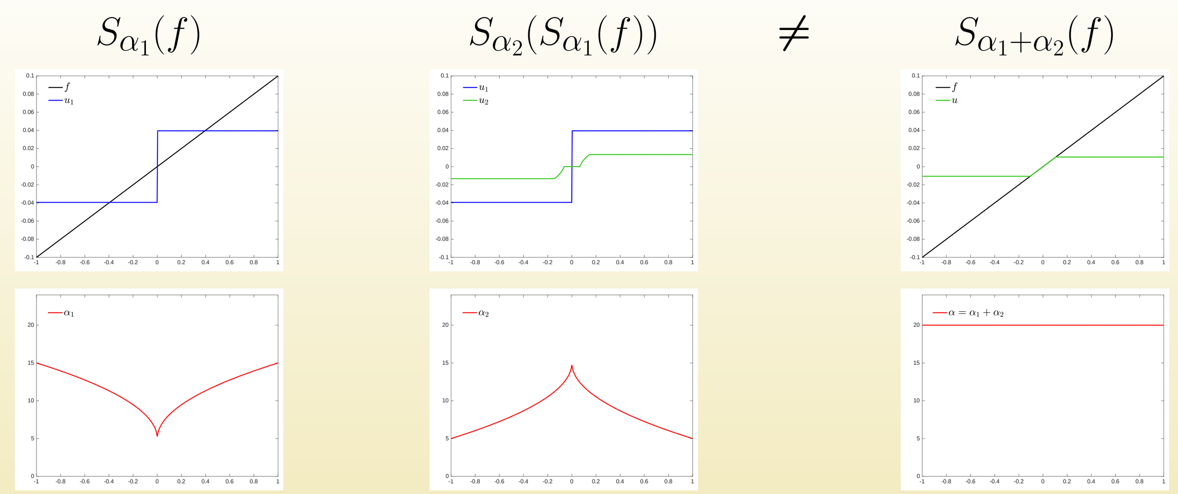
$$S_{\alpha_1+\alpha_2}(f)=S_{\alpha_2}(S_{\alpha_1}(f))=S_{\alpha_1}(S_{\alpha_2}(f)),\quad \alpha_1,\alpha_2\in\mathbb{R}^+.$$

In the weighted case this holds provided the second parameter is a scalar:

Partial Semigroup Property: Let $\Omega=(a,b)\subseteq\mathbb{R}$, $f\in \mathrm{BV}(\Omega)$, $\alpha_1\in C(\overline{\Omega})$ with $\alpha_1>0$ and $\alpha_2>0$ be a scalar. Then

$$S_{\alpha_1+\alpha_2}(f)=S_{\alpha_2}(S_{\alpha_1}(f)).$$

Counterexample when both weight parameters are not constants



Weighted TV - $|Du|(\Omega)\leq|Df|(\Omega)$

While it is immediate to show that for the solution u of the standard scalar TV minimisation, there holds $|Du|(\Omega)\leq|Df|(\Omega)$, an adaptation of the proof for the weighted case would only give $|Du|(\Omega)\leq\frac{\max\alpha}{\min\alpha}|Df|(\Omega)$. Using fine scale analysis of the structure of solutions of the weighted TV problem, we have shown that the same estimate holds for the weighted TV case as well:

Theorem: Let $\Omega=(a,b)\subseteq\mathbb{R}$, $f\in \mathrm{BV}(\Omega)$, $\alpha\in C(\overline{\Omega})$ with $\alpha>0$. Then if $u\in \mathrm{BV}(\Omega)$ solves the weighted TV problem with data f and weight α, it holds

$$|\boldsymbol{Du}|(\boldsymbol{\Omega})\leq|\boldsymbol{Df}|(\boldsymbol{\Omega}).$$

!! Counterintuitive, as weighted TV may create new jumps, increasing the variation locally.

Application: Provided $f\in \mathrm{BV}(\Omega)$, the weighted TV problem is well-posed even for vanishing weight function α, despite the lack of coercivity of the minimising functional:

Proposition: Let $\Omega=(a,b)\subseteq\mathbb{R}$, $f\in \mathrm{BV}(\Omega)$, $\alpha\in C(\overline{\Omega})$ with $\boldsymbol{\alpha\geq 0}$. Then the problem

$$\min_{u\in \mathrm{BV}(\Omega)} \frac{1}{2}\int_\Omega (f-u)^2dx + \int_\Omega \alpha(x) d|Du|$$

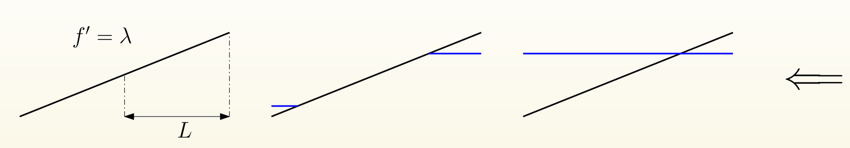
has a unique solution in $\mathrm{BV}(\Omega)$.
Proof by using a Γ-convergence argument and the estimate $|Du|(\Omega)\leq|Df|(\Omega)$.

Weighted fidelity model

The structure of solutions of the weighted fidelity model is simpler and resembles more the one of the scalar case:

- There holds $Du=0$ in the areas when $f\neq u$, in contrast to the weighted TV case.
- If $w\geq 0$ then $J_u\subseteq J_f$, i.e., new discontinuities cannot be created, again unlike the weighted TV case.

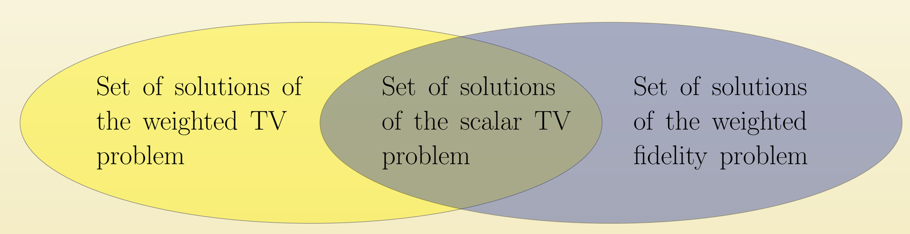
How different can the set of solutions of the two models be?



⟸ These are all the possible types of solutions for the weighted fidelity problem with data $f(x)=\lambda x$ and for **any** weight function w.

Proposition: The weighted TV problem with data $f(x)=\lambda x$ cannot produce any solution of the above type, apart from the symmetric ones, i.e., the ones that are produced by the scalar TV minimisation.

In other words, regardless of the weights α and w, for this specific affine data function we have:



➡ The two models can be very different even for simple data functions!

References

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