In image processing, non-local models are capable of exploiting information from far away pixels and repeating patterns as well as preserving textures.

In functional analysis, non-local functionals have been introduced and shown to approximate classical local differential operators as the non-locality vanishes and in certain cases they also provide non-local characterisations of Sobolev and BV spaces. In this work, we study two versions of non-local Hessian functionals, the explicit and the implicit formulation. For the former one, we prove localisation results and use it to provide some novel derivative-free characterisations of higher order Sobolev and BV spaces while we apply the latter one in image denoising with very promising results.

### Non-local Hessian—The explicit formulation

We define the explicit non-local Hessian \( H_u \) of a function \( u : \mathbb{R}^N \to \mathbb{R} \) at a point \( x \in \mathbb{R}^N \) as follows:

\[
H_u(x) := \frac{N(N+2)}{2} \int_{\mathbb{R}^N} \frac{u(x+h) - 2u(x) + u(x-h)}{|h|^2} \left( \frac{h \otimes h - \frac{h h^T}{|h|^2} \rho_h(h) \right) dh, \tag{1}
\]

where \( I_N \) is the \( N \times N \) identity matrix and \( \rho_h : \mathbb{R}^N \to \mathbb{R} \) are radially symmetric weight functions that satisfy

\[
\rho_h \geq 0, \quad \lim_{|h| \to 0} \rho_h(h) = 1, \quad \lim_{|h| \to \infty} \rho_h(h) = 0, \quad \forall \gamma > 0. \tag{2}
\]

The integral (1) (that can be defined more rigorously as a distribution) exists and indeed for functions with at least BV\(^2(\mathbb{R}^N)\) regularity where BV\(^2(\mathbb{R}^N) := \{ u \in W^{2,1}(\mathbb{R}^N) : \nabla u \in BV(\mathbb{R}^N, \mathbb{R}^N) \} \). Moreover, the following localisations hold:

**Theorem: Localisation results for \( H_u \)**

Let \( \rho_h \) be a sequence as above. Then the following convergences hold as \( n \to \infty \):

(i) If \( u \in C^2(\mathbb{R}^N) \) then \( H_u \to \nabla^2 u \) uniformly.

(ii) If \( u \in W^{2,p}(\mathbb{R}^N), 1 < p < \infty \) then \( H_u \to \nabla^2 u \) in \( L^p(\mathbb{R}^N, \mathbb{R}^N) \).

(iii) If \( u \in BV(\mathbb{R}^N) \) then \( H_u \rightharpoonup D^2 u \) in measure (strictly if \( N = 1 \), where \( D^2 u = D(\nabla u) \) is here a \( \mathbb{R}^{N \times N} \)-valued finite Radon measure).

Using the explicit formulation of non-local Hessian we obtain some novel derivative-free characterisations of the spaces \( W^{2,p}(\mathbb{R}^N) \) and \( BV(\mathbb{R}^N) \) as follows.

**Theorem: Non-local characterisations of \( W^{2,p}(\mathbb{R}^N) \) and \( BV^2(\mathbb{R}^N) \)**

(i) Let \( u \in L^p(\mathbb{R}^N), 1 < p < \infty \). Then

\[
W^{2,p}(\mathbb{R}^N) \iff \liminf_{n \to \infty} \| H_u \|_{L^p} < \infty.
\]

(ii) Let \( u \in L^1(\mathbb{R}^N) \). Then

\[
W^{2,1}(\mathbb{R}^N) \iff \liminf_{n \to \infty} \| H_u \|_{L^1} < \infty.
\]

### Non-local Hessian—The implicit formulation

We define an implicit formulation of a non-local Hessian which is more versatile than (1) and allows for non-symmetric weights, thus making it more suitable for imaging applications. We define the explicit non-local gradient and non-local Hessian, \( G_u(x) \) and \( H_u(x) \) respectively as

\[
(G_u(x), H_u(x)) := \arg\min_{G \in \mathcal{C}_u, \mathcal{H} \in \mathcal{B}_u} \int_{\mathcal{D} \times \{x\}} \left( u(x+h) - u(x) - G(h) \frac{h}{|h|} \right) \rho_h(h) dh, \tag{3}
\]

Since the optimality conditions for (3) are linear we can use \( H_u \) as a regulariser as follows:

**Non-local Hessian—The implicit formulation**

\[
\min_{s = 1, 2} \frac{1}{2} \| u - f \|^2_s + \alpha \| H_u \|_s, \quad s = 1, 2, \quad \text{subject to (3)}.
\]

**Convex objective + linear constraints \( \implies \) Solvable with standard convex solvers.**

We choose the weight function \( \rho_h \) in a way that it is small for points that are separated from \( x \) by a strong edge and large otherwise. This is done by solving the Eikonal equation

\[
\nabla c_r(y) = |\nabla f|, \quad c_r(x) = 0, \quad \phi(\nabla c_r) = \| \nabla f \|^2_s + \epsilon,
\]

and setting

\[
\rho_h(y) = \begin{cases} \frac{1}{\epsilon} |\nabla c_r(y)|^2 & \text{if } i \leq K, \\ 0 & \text{if } i > K,
\end{cases}
\]

for the \( K \) closest neighbours of \( x \) (in the \( \epsilon \)-distance sense).

\( \implies \) we obtain a regulariser that preserves edges and affine structures as well.

**Theorem: Connection between the explicit and implicit non-local Hessian**

Assume that \( \rho_h = \rho \) is radially symmetric. Then the non-local Hessian \( H_u \) can be written as

\[
H_u = \int_{\mathbb{R}^N} H_u^i(h) \int_{\mathbb{R}^N} \rho^i(z) d\Gamma^{N-1}(z) dh.
\]

We are able to obtain true piecewise affine reconstructions with minimal loss of contrast.

**Some denoising examples**

**First row:** Denoising results. **Second row:** Middle row slices. **Third row:** Distribution of the \( L^2 \) error to the ground truth. The parameters are chosen for optimal PSNR.

**Input, PSNR=22.82, TV, PSNR=31.83, L^2-TGV, PSNR=33.28, L^2-H^1, PSNR=35.39, L^2-H^2, PSNR=36.06**

**References**


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