Analytical Solutions to the One Dimensional L^2–TGV^2 Problem

Kostas Papafitsoros [1] & Kristian Bredies [2]
1. Cambridge Image Analysis, Cambridge Centre for Analysis, University of Cambridge, UK
2. University of Graz, Austria.

Image denoising

Image denoising denotes the task of removing (random) noise from a digital image. While doing so, it is also desirable to **preserve** certain characteristics of the image like **edges**, **contrast** and **smooth structures**


Noisy image


TGV denoised image

Total Generalised Variation

In the variational approach one obtains the reconstructed image as a minimiser of a problem of the following kind:

$$\min_{u\in\mathcal{B}} \underbrace{\frac{1}{2}\int_\Omega (u-f)^2 dx}_{\text{fidelity term}} + \underbrace{\Psi(u)}_{\text{regulariser}} .$$

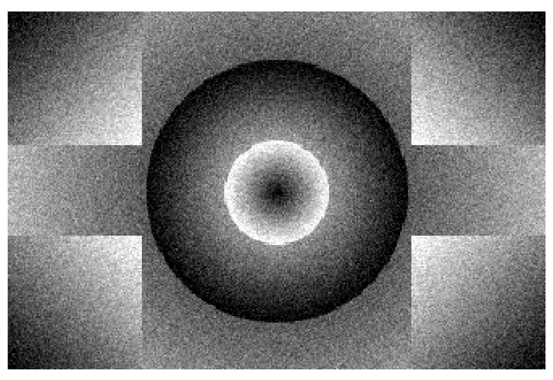
Here, Ω is typically a rectangle, f are the noisy data and $\mathcal{B}$ is an appropriate function space defined on Ω. The regulariser is responsible for the denoising process. A quite popular choice for Ψ has been the total variation weighted with a parameter $\alpha > 0$:

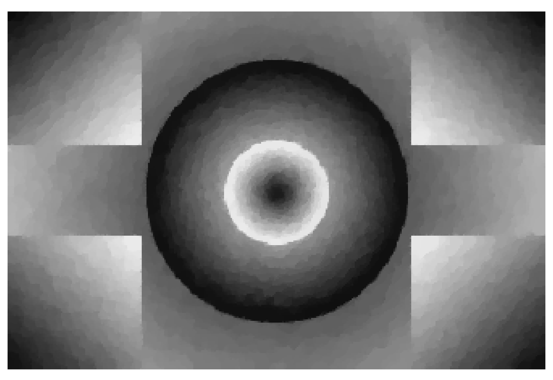
$$\alpha\mathrm{TV}(u) := \alpha \sup\left\{\int_\Omega u \operatorname{div} v\, dx :\ v \in C_c^1(\Omega,\mathbb{R}^2),\ \|v\|_\infty \le 1\right\} = \alpha|Du|(\Omega).$$

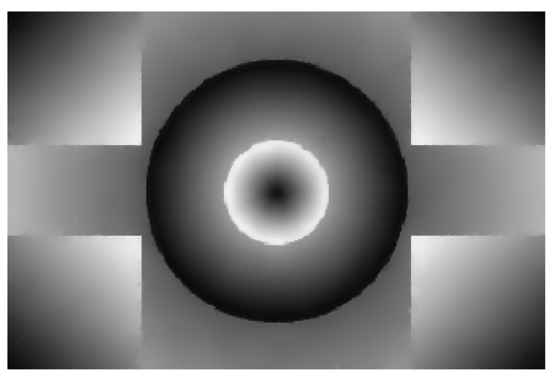
TV minimisation is an edge preserving method but also results in *blocky-like* images since it promotes piecewise constant reconstructions.
The second order total generalised variation (TGV^2) with parameters α and β is defined as

$$\mathrm{TGV}^2_{\beta,\alpha}(u) := \sup\left\{\int_\Omega u \operatorname{div}^2 v\, dx :\ v\in C_c^2(\Omega, S^{2\times2}),\ \|v\|_\infty \le \beta,\ \|\operatorname{div} v\|_\infty \le \alpha\right\} = \min_{w\in \mathrm{BD}(\Omega)} \alpha|Du-w|(\Omega) + \beta|Dw|(\Omega),$$

where $S^{2\times2}$ is the set of the 2×2 symmetric matrices and $\mathrm{BD}(\Omega)$ is the space of functions of bounded deformation. TGV^2 is a high quality regulariser that has the ability to adapt to the regularity of the data by optimally balancing first and second order derivatives, thus preserving edges and smooth structures as well:


Noisy image


TV denoised image


TGV^2 denoised image

In order to understand deeper how this kind of regularisation behaves, we compute analytical solutions for simple data functions in dimension one and we investigate how these solutions change with respect to the parameters α and β.

The optimality conditions

In dimension one, the TGV^2 denoising minimisation problem reads as follows:

$$\min_{\substack{u\in \mathrm{BV}(\Omega)\\ w\in\mathrm{BV}(\Omega)}} \frac{1}{2}\int_\Omega (u-f)^2 dx + \alpha|Du-w|(\Omega) + \beta|Dw|(\Omega). \qquad (P)$$

where $\mathrm{BV}(\Omega)$ is the space of functions of bounded variation. Using Fenchel–Rockafellar duality we derive the following theorem:

Theorem – Optimality conditions for (P)

A pair $(u,w)\in \mathrm{BV}(\Omega)\times\mathrm{BV}(\Omega)$ is a minimiser for (P) if and only if there exists a function $v\in H_0^2(\Omega)$ such that

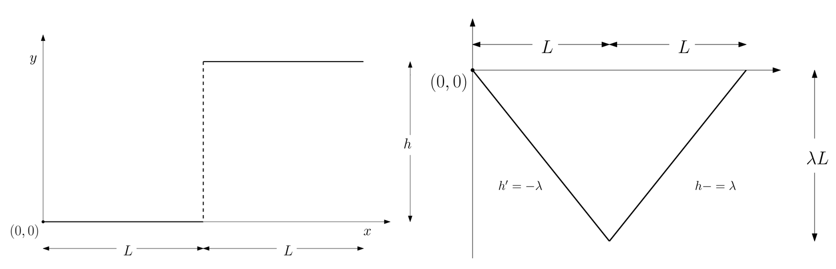
$$v'' = f-u,$$
$$-v' \in \alpha\,\mathrm{Sgn}(Du-w),$$
$$v \in \beta\,\mathrm{Sgn}(Dw),$$

where for a Radon measure $\mu = \mathrm{sgn}(\mu)|\mu|$ we define

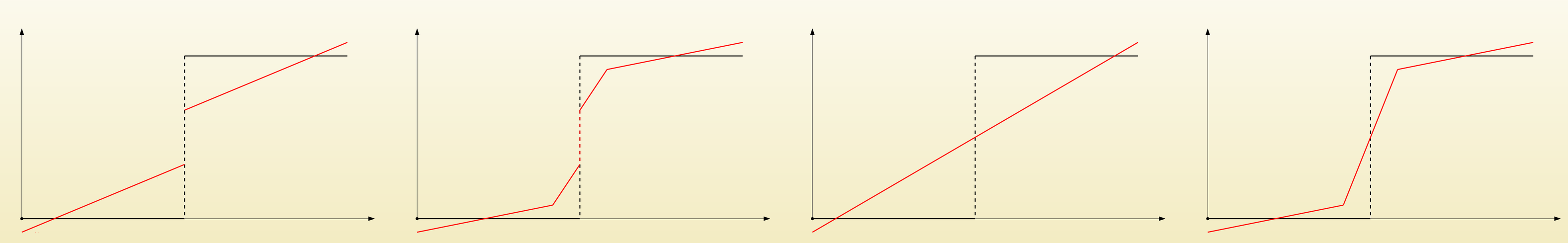
$$\mathrm{Sgn}(\mu) = \{v\in L^\infty(\Omega)\cap L^\infty(\Omega,|\mu|):\ \|v\|_\infty \le 1,\ v=\mathrm{sgn}(\mu),\ |\mu|-\text{a.e.}\}.$$

Analytical solutions

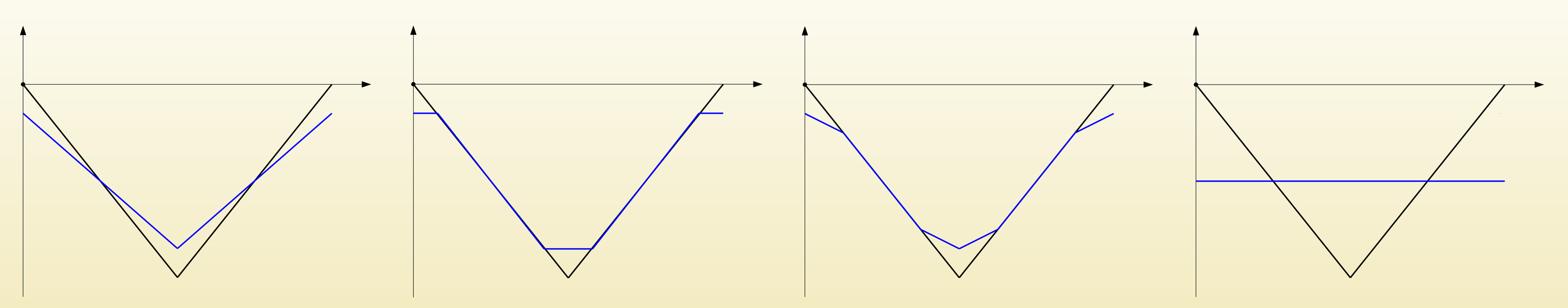
Using the above optimality conditions we compute analytical solutions for a simple jump and hat function:



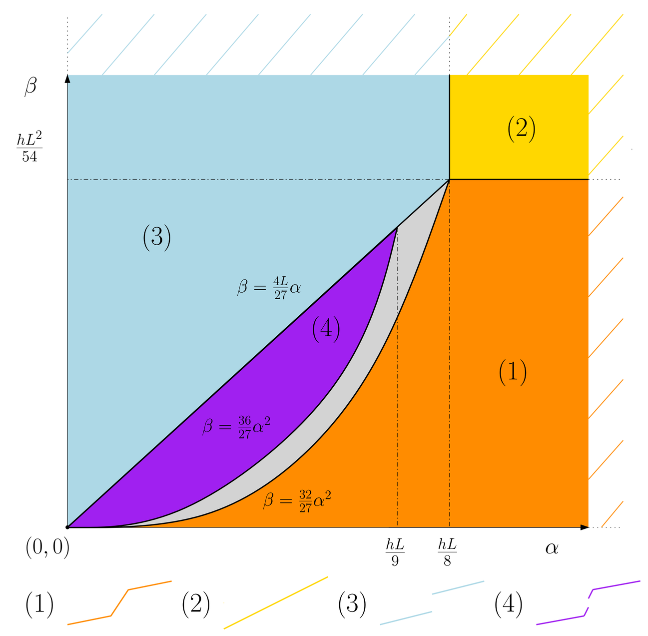
The 4 possible solutions for the jump function

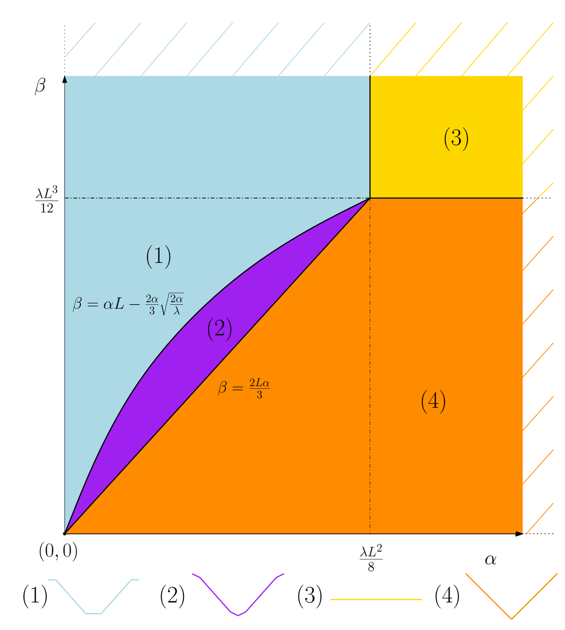


The 4 possible solutions for the hat function



We are also able to tell for which combinations of the parameters α and β we have each solution:





References

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